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To cite this article:

Maher, E. M., & He, J. (2023). Supporting spatial reasoning: Identifying aspects of length, area, and volume in textbook definitions. *International Journal on Studies in Education (IJonSE), 5*(1), 64-87. https://doi.org/10.46328/ijonse.100

International Journal on Studies in Education

2023, Vol. 5, No. 1, 64-87 https://doi.org/10.46328/ijonse.100

Supporting Spatial Reasoning: Identifying Aspects of Length, Area, and Volume in Textbook Definitions

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Article Info

Article History

Received:
29 March 2022
Accepted:
28 August 2022

Keywords

Preservice teachers
Definitions
Spatial measurement
Textbook analysis

Abstract

Length, area, and volume share structural similarities enabling flexibility in reasoning for real-world applications. Deep understanding of structures can help teachers connect these concepts to support their students' mathematical reasoning and practices involving real-world situations. In mathematics textbooks designed for future teachers, definitions of length, area, and volume vary from *procedural* (e.g., use a ruler to measure side lengths, use formulas to calculate measures) to *conceptual* (e.g., construct appropriate *n*-dimensional units that tessellate the *n*-dimensional space) to *formal* (e.g., construct a function mapping qualitative size to a quantity of appropriate units). Most textbooks describe length, area, and volume as quantitative measurements and provide examples of standard units. Definitional aspects such as describing size as an attribute or measurement, identifying dimensionality of a space, or constructing appropriate nonstandard units are inconsistently acknowledged across textbooks. Attending to definitional aspects of spatial attributes and their quantification can open conversations about the structure and essential meanings of length, area, and volume.

Introduction

Length, area, and volume share structural similarities that can be connected to enable flexible reasoning in application of these concepts. Teachers who deeply understand these structural similarities can also connect them to support their elementary students' reasoning and practices involving measurement of the real world. Further, spatial reasoning and measurement-related practices connect directly to other essential mathematical tools and topics such as fractional reasoning, number line reasoning, and algebraic reasoning (Georgia Department of Education, 2021; Lee, 2012; National Council of Teachers of Mathematics, 2000). Research shows that weakness in spatial measurement understandings correlates to weakness in these related areas and develops from misconceptions about the meaning of measurement and how memorized formulas relate to and constructed on physical measurement processes (e.g., Kim, et. al., 2019).

Struggles with spatial measurement (e.g., length, area, and volume) are seen nationally and internationally. The United States' National Assessment of Educational Progress (NAEP) shows persistent struggles in the United States (e.g., Kloosterman et al., 2009). The 2015 Trends in International Mathematics and Science Study (TIMSS) confirms this struggle in U.S. students and students in other countries (e.g., Mullis et al., 2016). Graduating K-8 preservice teachers (PSTs) often share their future students' struggles. For example, understanding relationships between measures of area and perimeter are difficult for students (Bamberger & Oberdorf, 2010; Barrett & Clements, 2003; Tan Sisman & Aksu, 2016) and their future teachers (Livy et al., 2012; Wanner, 2019).

Prospective teachers experience spatial reasoning and measurement-related practices, along with many other topics, in mathematics content courses required as a part of their preparation for teaching. Tasks and experiences, in classroom interactions and in homework completed outside the classroom, impact prospective teachers' informal reasoning about spatial measurement concepts (e.g., Kabael & Akin, 2018; Oliveira & Henriques, 2021; Ozturk, Demir, & Akkan, 2021; Yackel & Cobb, 1996). This study focuses on textbooks written for these mathematics content courses because, even though implementation varies across classrooms (Tarr et al., 2008), textbooks play an important role in what teachers teach, the opportunities they present for mathematical sense-making (Stein et al., 2007), and their profound impact on K-12 and university classroom instruction (McCrory et al., 2009). That is, as future teachers engage with their mathematics textbooks (whether physical or online), they develop strategies for reading and using textbooks in their own learning that could impact their future decision-making as teachers. Whether formal or informal, creating and analyzing definitions helps deconstruct the structures and meanings that make up mathematical concepts.

We analyzed textbooks commonly used in United States mathematics content courses for prospective K-8 teachers, based on a list compiled by McCrory (2006). A few books are written by mathematicians while the others are written by mathematics education researchers from a variety of backgrounds. Here we build on an updated version of McCrory's (2006) textbook list (shown in the Appendix). Definitions and explanations of length, area, and volume topics vary widely across these textbooks. To analyze similarities and differences, we build on a framework of definition characteristics (Gilbertson et al., 2016) described in depth below. The framework is not a comprehensive list of aspects of spatial concepts; rather, it is a lens used to compare definitions of length, area, and volume as they are presented in mathematics content textbooks designed for future teachers. We organize Gilbertson's et al. (2016) framework within the four-step process of measurement (e.g., Long & DeTemple, 2012; Stehr & He, 2019): (1) select an object (and an attribute of the object) to measure; (2) select a unit of measure; (3) compare the object's attribute with the unit of measure; and (4) express or communicate the measure. We asked the following questions in this study: *What characteristics of definitions are found in elementary mathematics textbooks across spatial measurement topics (i.e., length, area, and volume)? What similarities or differences exist across textbooks in the definitions of these concepts?*

Method

Although classroom instruction of mathematics content courses varies widely in teacher preparation programs, textbooks "define a substantial element of what students have an *opportunity* to learn" (McCrory, 2006, p. 20). Definitions in mathematics textbooks are often typeset to stand out from the other text, and future teachers may write the definitions in notes or memorize them using notecards. Even when future teachers do not see them, we argue that definitions are still valuable objects of our attention. Definitions are designed by mathematicians as a formal way to communicate essential aspects of a concept. Instructors of mathematics content courses for future teachers may therefore bring the textbook definitions into their teaching. In that case, future teachers may still see definitions as summaries of the concepts. Our selection of textbooks is guided by McCrory (2006) and represents a wide range of textbooks that vary in the organization, coverage of the topics, and pedagogy. The books are written by mathematicians, mathematics educators, or both. We examined definitions related to length, area, and volume/capacity in 11 textbooks specifically developed to be used with future elementary teachers in their mathematics content courses. The textbooks and all definitions are listed in the Appendix.

Procedures

Two researchers adapted an existing framework to code definitions of length, area, and volume in textbooks designed for teaching 5-11 years old students (Gilbertson et al., 2016). The coding unit, a definition, is usually a statement that describes the meaning of a term (e.g., length, area, volume). Textbooks may have included other explanatory text and tasks, but we looked for text that was set apart and written as a summary description in a way that designates a definition. Two researchers independently coded the content, compared the coding, and resolved discrepancies. In the next section, we provide our analytical frameworks and findings along with coding examples.

Analytical Framework and Findings

We provide our analytical frameworks in the following subsections organized by four steps of a measurement process. At each step, we share our coding framework organized by aspects, description of the aspect, and questions that guided our coding. We share overall frequencies for the aspects of that step. We then share examples of length, area, and volume as exemplars of our coding and meaning of the aspects. We choose the exemplars to vary across textbooks and to include the different aspects. First, we focus on selection of *what* to measure (i.e., an object or attribute of the object). We next consider selection of the *unit* of measure (i.e., standard units such as centimeters or nonstandard units such as tiles). Third, we share our framework for *quantifying* the measurement (i.e., comparing the attribute to the unit of measurement) and, fourth, *expressing* that measurement.

Step 1: Select Object (and Attribute of Object) to Measure

In a definition of length, area, or volume, the following common aspects may be included and are shown in Table 1: the concept as an *attribute* (i.e., the size of a space) and examples of other names used for that attribute; the concept as a *measure* of the attribute; the mention of *space*; the mention of *dimensionality*; and indication of the need for *boundaries* or mention of examples of specific named boundaries (i.e., a circle bounds a two dimensional space). A measurable attribute is a characteristic of an object that can be quantified. The *attribute* of an object and the *measure* of that attribute are typically described using the same word; for example, the *length* of an object refers to the *attribute* (i.e., the one-dimensional space that exists between endpoints) and to the *measure* (i.e., quantification of that space). *Examples* of different names for an attribute are mostly found for length, where special types of length can include height, width, and depth. The undefined concept of *space* refers to a continuous region that makes up the size of an attribute of an object. To measure a space, a *boundary* must be indicated. Speaking formally and precisely, boundaries for spaces could be: a pair of endpoints of a curve or path for length, closed curve for area, and closed figure for volume. A specific boundary enclosing an area could be a circle, and a volume could be an octahedron.

Table 1. Step 1 Coding Aspects

Aspects	Descriptions of Aspect	Coding Questions
Attribute	concept as one-, two-, or three-dimensional space and named examples, e.g., height, width, depth are types of length	Is a concept mentioned as an attribute, measure, or both? Are examples of different names for an attribute given?
Measure of Attribute	concept as the quantification of the spatial attribute (i.e., expression in units)	
Space	undefined term meaning a 1D, 2D, or 3D attribute, the size of which may be measured	Is the word "space" explicitly stated?
Dimensionality	number of independent directions or lengths along which to move within a given space	Is dimensionality mentioned? If yes, which?
Boundary	endpoints (1D), simple closed curve (2D), or simple closed figure (3D) enclosing a space; examples of named boundaries are, e.g., circles are boundaries of areas	Is there mention of endpoints or other enclosure? Or does it mention a specific shape or object that has a clear boundary?

The authors examined each definition related to length, area, and volume/capacity in the selected 11 textbooks (see the Appendix) for mention of the aspects shown in Table 1. Some textbooks did not include a definition, while others included multiple definitions. In Table 2, the final counts are shown. More than half of the definitions included a description of the concept as a measure. Half of the length and area definitions mentioned dimensionality, while over half of the area and volume definitions mentioned boundaries. We share examples of definitions and our coding of the definitions below, organized by length, area, and volume.

Table 2. Step 1 Coding Frequencies

Aspects	Length (10)	Area (16)	Volume / Capacity (23)
Concept as attribute	4 (40%)	7 (44%)	11 (48%)
Concept as a measure / quantification	9 (90%)	15 (94%)	15 (65%)
Space	0 (0%)	1 (6%)	8 (35%)
Dimensionality	5 (50%)	8 (50%)	9 (39%)
Boundary	4 (40%)	8 (50%)	14 (61%)

Coding Length Definitions

In Table 3, three definitions of length, selected to fully illustrate each aspect, are shown with their analyses. The aspects found in a particular definition are shown in the column to the right of the definition. Each of the first two examples clearly describes the two meanings of length: as a measure and as an attribute. The third example represents distance as a length and a number, without acknowledging the two uses of the word. For example, Beckmann (2011) describes length as an *attribute*, "the size of something," and as a *measure of the attribute*, "length is … how many of a chosen unit of length" (p. 481). Sowder et al. (2010) also described length as a "quality or attribute" and as "the measurement of that quality" (p. 528).

Table 3. Step 1 Length Examples

Definitions (emphasis added)	Aspects of Length Definition
*A length describes the **size** of something (or a part of something) that is **one-dimensional**; the length of that one-dimensional object **is how many** of a chosen unit of length (such as inches, centimeters, etc.) it takes to cover the object without gaps or overlaps, where it is understood that we may use parts of a unit, too. Roughly speaking, an object is **one-dimensional** if at each location, there is only one independent direction along which to move within the object.* (Beckmann, 2011, p. 481)	Attribute; Measure; Dimension (+ definition)
*We speak of the length of a **piece of wire or a rectangle** in two ways. The term length might refer to the **quality or attribute** we are focusing on, or it might refer to the **measurement** of that quality. The context usually makes clear which reference is intended.* (Sowder et al., 2010, p. 528)	Attribute; Measure; Boundary
*In geometry one also has a notion of length and distance. The **distance between two points** A and B is the length of the segment $\underline{AB}$; in this book we will denote this length by AB (some textbooks use a different notation for length). Note that **AB is a number** whereas **$\underline{AB}$ is a segment**.* (Parker & Baldridge, 2008, p. 5)	Measure; Boundary

Space was not mentioned in any length definition. The first definition mentions dimensionality and defines it. The other definitions indicate boundaries in different ways. Sowder et al. (2010) indirectly includes boundaries by giving examples of bounded objects such as a piece of wire or a rectangle, while Parker and Baldridge (2008) explicitly mention endpoints. One interesting element of the Sowder et al. definition is that it mentions lengths of three-dimensional and two-dimensional objects without noting that, because the objects have more than one dimension, each has multiple choices for length (e.g., size of a rectangle's height, width, diagonal, or perimeter).

Coding Area Definitions

In Table 4, definitions of area from three textbooks are shown with their analyses. Definitions for these examples were chosen for their differences to fully illustrate each aspect and how it may appear in a definition. The first and last examples clearly describe area as an attribute while the first two examples describe area as a measure. Bennett et al. (2012) describes area as the "sizes of … surfaces" and as the "number of units it takes to cover a surface" (p. 676). Van de Walle et al. (2013) explains it as "the two dimensional space inside a region" (p. 384).

Table 4. Step 1 Area Examples

Definitions (emphasis added)	Aspects of Area Definition
*To measure the **sizes** of **plots of land, panes of glass, floors, walls, and other such surfaces**, we need a new type of unit, one that can be used to cover a surface. The **number of units** it takes to cover a surface is called its area.* (Bennett et al., 2012, p. 676)	Attribute & Measure; Boundary + Examples
*Area is a **measure** of the region bounded by a **closed plane curve**.* (Long & DeTemple, 2012, p. 530)	Measure only; Dimension; Boundary
*Area is the **two dimensional space** inside a region.* (Van de Walle et al., 2013, p. 384)	Attribute only; Space; Dimension

The second and third examples include dimensionality implicitly or explicitly. For example, Long and Temple (2012) use the phrase "closed plane curve" which implicitly references the boundary of a two-dimensional space (p. 530). Van de Walle et al. (2013) explicitly mentioned dimensionality in the phrase "two-dimensional space" (p. 384). In this phrase, Van de Walle et al. also mentions space explicitly. Boundaries are implicitly and explicitly mentioned in Bennett et al. and Long and DeTemple, respectively, by using examples of bounded regions and by using the precise mathematical language of "a closed plane curve."

Coding Volume Definitions

In Table 5, definitions of volume from three textbooks are shown with their analyses. Definitions for these examples were chosen for their differences to fully illustrate each aspect and how it may appear in a definition. We include Musser et al. (2011) to highlight the difficulties in separating volume and capacity. Musser et al. explicitly separate the two attributes where capacity is "the amount [the vase] will hold" and volume is "the amount of material comprising the vase itself" (p. 680). The first and last examples clearly describe volume as an attribute while the last two examples describe volume as a measure. The third example explicitly uses the phrases "three-dimensional" and "cubic units" (cubic implies three-dimensionality). The first and third examples mention space explicitly in describing "amount of space" within an object or occupied by an object. Boundaries are explicitly and implicitly mentioned in examples one and two, respectively, by mentioning "within the object" and giving the examples of a vase and a water glass.

Table 5. Step 1 Volume Examples

Definitions (emphasis added)	Aspects of Volume Definition
amount of space *contained* ***within that object.*** (Bassarear, 2012, p. 637)	Attribute only; Space; Boundary
To ***measure*** *the capacity of water that a* ***vase*** *will hold, we can select a convenient container, such as a* ***water glass****, to use as our unit and count* ***how many glassfuls*** *are required to fill the vase. This is an informal method of measuring volume. (Strictly speaking, we are measuring the capacity of the vase, namely the amount that it will hold.) The volume of the vase would be the amount of material comprising the vase itself.* (Musser et al. 2011, p. 680)	Measure only; Boundary + Examples
Volume is the ***amount of space*** *occupied by a* ***three-dimensional*** *figure. Volume is usually* ***measured in cubic units****. Whereas surface area is the total area of the faces of a solid, volume is the* ***capacity of a solid****.* (Sonnabend, 2010, p. 571)	Measure & Attribute; Space; Dimension

Step 2: Select Unit of Measure

A unit of measure is any reproducible unit with the same dimensionality as the attribute being measured. *Standard* units are units that are commonly defined such as inches or meters for length, acres or square kilometers for area, and cups or milliliters or cubic centimeters for volume. A *nonstandard* unit is a unit that does not have a common definition but can be used for measurement. Nonstandard units have limitations when communicating measures but can still be useful. In particular, they are used flexibly in response to an immediate need when other tools are not available. For example, I can use the width of my thumb to measure part of my bicycle and compare the measure to a basket that I want to buy.

My thumb is a nonstandard unit that I understand and can replicate. I cannot expect to call a store, however, and expect a store employee to easily replicate the unit of “my thumb-width” to check on sizes of baskets for me. In choosing a unit, different *rationales* may be used, including convenience, ease of use, and precision. In the example of nonstandard units, the rationale for using my own thumb as a unit of length is based on convenience and ease of use.

However, communicating the length to someone with a different thumb size can be difficult. Using a thumb could likewise make precision difficult. Precision of measurement is impacted by the ease of including parts of a unit. A unit of measure may be *continuous* or *discrete*. For example, a unit of length could be the length of a tile iterated to fill (discrete) or a rubber band stretched to cover (continuous) a one-dimensional space. Table 6 summarizes these aspects of coding for Step 2.

Table 6. Step 2 Coding Aspects

Aspects	Descriptions of Aspect	Coding Questions
Units	any reproducible shape with the same dimensionality as the attribute being measured that can be used to cover or fill the attribute (i.e., the unit tessellates)	Is the term *unit* explicitly used? Is a *unit* defined? Are examples of units given?
Standard/Nonstandard	*standard* are precisely and conventionally defined; *nonstandard* units are not	Is the term *standard units* explicitly used? Is the term *nonstandard units* explicitly used?
Rationale	may include convenience, ease of use, precision	Is a rationale for choice of units indicated?
Parts of Units	units may be subdivided to provide higher precision of measure	Are parts or fractions of units mentioned?
Continuous/Discrete	*continuous* units can easily adapt to any level of precision versus *discrete* units which may be broken into parts of units but beyond halves or quarters the precision is lost	What type of substance/object fills the space being measured (e.g., a continuous quantity of liquid or a discrete number of linking cubes)?

The authors examined each definition for mention of the aspects shown in Table 6. In Table 7, the final counts are shown. Not all definitions mentioned units, so we included only 5 (50%) of length, 13 (81%) of area, and 12 (52%) of volume definitions. Of the definitions that mentioned units, most gave examples of discrete units. Most volume definitions explicitly mentioned standard units, and only a few length, area, or volume definitions mentioned nonstandard units. Note that Table 7 only includes definitions that mentioned units. We share examples of our coding below.

Table 7. Step 2 Coding Frequencies

Aspects	Length (5)	Area (13)	Volume / Capacity (12)
Definition of unit	1 (20%)	8 (62%)	2 (17%)
Examples of units	2 (40%)	12 (92%)	10 (83%)
Standard units	2 (40%)	2 (13%)	11 (92%)
Nonstandard units	1 (20%)	2 (13%)	3 (13%)
Rationale for Unit Choice	0 (0%)	3 (19%)	2 (17%)
Parts or fractions of a unit	1 (20%)	2 (13%)	1 (8%)
Continuous units	0 (0%)	2 (13%)	4 (33%)
Discrete units	4 (80%)	13 (100%)	10 (83%)

Coding Length Definitions

Five (half) of the length definitions mentioned units explicitly. Four of the five focused on discrete length units. In Table 8, definitions of length from two textbooks are shown with their analyses. Only aspects with a frequency greater than zero are shown in the table; continuous units and a rationale for unit choice did not appear in any length definition. Definitions for these examples were chosen for their differences to fully illustrate each aspect and how it may appear in a definition.

Table 8. Step 2 Length Examples

Definitions (emphasis added)	Aspects of Length Definition
A length describes the size of something (or a part of something) that is one-dimensional; the length of that one-dimensional object is how many of a ***chosen unit of length*** *(such as* ***inches, centimeters,*** *etc.) it takes to cover the object without gaps or overlaps, where it is understood that we may use* ***parts of a unit, too****.* (Beckmann, 2011, p. 481)	Unit Examples; Standard only; Parts of Units; Discrete Units
There are four important principles of iterating ***units of length****, whether they are* ***nonstandard*** *or* ***standard****: *****All units must have equal length*** *- if not, you cannot accumulate units by counting. *****All units must be placed on the path*** *being measured - otherwise, a different quantity is being measured. *The* ***units must be without gaps*** *- if not, part of the quantity is not being measured. *The* ***units must not overlap*** *- otherwise, part of the quantity is measured more than once.* (Van de Walle et al., 2013, p. 382)	Unit Definition; Standard and Nonstandard; Discrete Units

The example definitions shown in Table 8 highlight various aspects of the use of *unit* in length definitions. Only Van de Walle et al. (2013) attempted to define a length unit, although it is not explicitly stated. We coded it as a definition of length unit because Van de Walle et al. mentions that “all units must have equal length” and are placed without gaps or overlaps suggesting the length unit has many identical copies and can *tessellate* linear space (e.g., be placed without gaps or overlaps). Beckmann (2011) gives examples of standard length units as inches and centimeters, while Van de Walle acknowledges length units might be standard or nonstandard (but does not give examples of nonstandard units). Beckmann explains that parts of units may be used, indicating that measurements may be fractional. Both definitions describe units as discrete lengths that are iterated.

Coding Area Definitions

Of the 16 area definitions, 13 mentioned units explicitly. All definitions that mentioned units used descriptions that indicated discrete units, although two also described continuous units. All but one definition also included examples of units. Definitions of area with their analyses are shown in Table 9. These examples were chosen to illustrate a variety of aspects of coding.

Table 9. Step 2 Area Examples

Definitions (emphasis added)	Aspects of Area Definition
*Questions about area generally deal with "how much" it takes to cover an object - for example, **how much fertilizer to cover a lawn**, **how much material to cover a bed**. In order to answer area questions, we have to **select an appropriate unit**, and thus the answer takes the form of **how many of those units**.* (Bassarear, 2012, p. 621)	Unit Examples; Continuous and Discrete Units
*Students recognize area as an attribute of two-dimensional regions. They learn that they can quantify area by finding the total number of **same-sized units of area** that cover the shape without gaps or overlaps. They understand that **a square unit that is 1 unit on a side is the standard unit for measuring area**. ...Area is measured using **square units** and the area of a region is the number of **nonoverlapping square units** that covers the region.* (Billstein et al., 2010, p. 854)	Unit Definition and Examples; Standard only; Discrete Units
*Area is a measure of the region bounded by a closed plane curve. **Any shape** could be **chosen as a unit**, but the **square is the most common**. The **size of the square is arbitrary, but it is natural to choose the length of a side to correspond to a unit measure of length**. Areas are therefore usually measured in **square inches, square feet**, and so on.* (Long & DeTemple., 2012, p. 530)	Unit Examples; Standard and Nonstandard; Rationale; Discrete Units

Billstein et al. (2010) defines an area unit as a set of same-sized areas that tessellate the two-dimensional space (e.g., cover without gaps or overlaps) and further define a standard unit as a square where a side is one length unit. Long and DeTemple (2012) acknowledged that both standard and nonstandard units may be used, but rationalized the use of squares because “it is the most common” and the choice of size because “it is natural to choose the length of a side to correspond to a unit measure of length” (p. 530). All three examples described discrete area units that can be counted; Long and DeTemple gave square inches and square feet as examples. Bassarear (2012) suggested the idea of continuous measure of area by describing a quantity of fertilizer or material.

Coding Volume Definitions

Slightly more than half of the volume definitions (12 out of 23) mentioned units explicitly. All but one definition that mentioned units also referenced standard units and all but two described discrete units and provided examples. Table 10 shows examples of volume definitions with their analyses. The definitions are chosen as examples of how different aspects of definitions were coded.

Billstein et al. (2010) defines a unit of volume as a “shape that tessellates space” and a standard cubic unit as “the amount of space enclosed within a cube that measures 1 [length] unit on a side” (p. 906). Musser et al. (2011)

explicitly gives an example of a nonstandard and continuous unit of volume (e.g., convenient container that holds water) to measure the interior volume of a vase. Musser et al. also indicate the potential confusion between describing a measure of volume (typically describes the space an object occupies) and of capacity (a type of volume measuring the internal volume of an object).

Table 10. Step 2 Volume Examples

Definitions (emphasis added)	Aspects of Volume Definition
To measure the capacity of water that a vase will hold, we can ***select a convenient container, such as a water glass, to use as our unit*** *and count how many glassfuls are required to fill the vase. This is an informal method of measuring volume. (Strictly speaking, we are measuring the capacity of the vase, namely the amount that it will hold).* (Musser et al., 2011, p. 680)	Unit Examples; Nonstandard only; Continuous Units
Volume describes how much space a three-dimensional figure contains. The ***unit*** *of measure for volume must be a* ***shape that tessellates space****.* ***Cubes tessellate space****; that is, they can be* ***stacked*** *so that they* ***leave no gaps and fill space****.* ***Standard units*** *of volume are* ***based on cubes*** *and* ***are cubic units****. A cubic unit is the* ***amount of space enclosed within a cube that measures 1 unit on a side****.* (Billstein et al., 2010, p. 906)	Unit Definition and Examples; Standard only; Rationale; Discrete Units
A volume describes the size of an object (or a part of an object) that is three-dimensional; the volume of that three-dimensional object is how many of a ***chosen unit of volume (such as cubic inches, cubic centimeters, etc.)*** *it takes to fill the object without gaps or overlaps, where it is understood that we may use* ***parts of a unit, too****.* (Bennett et al., 2012, p. 701)	Unit Examples; Standard only; Parts of Units; Discrete Units

Step 3 and Step 4: Finding and Expressing the Measure

Step 3 involves comparing an object's chosen attribute with an appropriate unit of measure. To measure an attribute such as length, area, or volume means to communicate its size in some way, which often means describing it in terms of small, identical units (i.e., lengths, areas, or volumes, respectively). The attribute is compared to a unit to determine the quantity of units (or parts of units) that *exhaust the space* being measured; that is, cover, fill, or otherwise complete the comparison of attribute and unit. Often, a measurement device is used and this *procedural tool use* can be described step by step (i.e., line one end of an object against the "0" on a ruler, and the measure will be the number on the ruler closest to the opposite end of the object).

Step 4 is to express or communicate the measure. Finalizing a measurement means *quantifying* the size of the attribute in terms of multiples of the standard or nonstandard unit or stating the size using other understandable language or comparison. The authors examined each definition for mention of the aspects shown in Table 11.

Table 11. Steps 3-4 Coding Aspects

Aspects	Descriptions of Aspect	Coding Questions
Exhaust Space	cover, fill, or otherwise complete the comparison of attribute and unit without leaving gaps or overlaps	Is there mention of the need to completely fill, cover or tile the space?
Procedural Tool Use	a measurement device and process described step by step	Is a specific measurement method using a tool and/or unit indicated?
Quantification	expressing the measure as multiples of the standard or nonstandard unit	Is there mention of counting, layering, reading a tool or applying a formula?

In Table 12, the final counts are shown. We share examples of our coding below. A description of exhausting and quantifying a space showed up in most area definitions, but less than half of length and volume/capacity definitions. Note that procedural tool use only appeared in length definitions; hence we do not include this aspect in our presentation of examples of area and volume definitions shown in Tables 14 and 15.

Table 12. Steps 3-4 Coding Frequencies

Aspects	Length (10)	Area (16)	Volume / Capacity (23)
Exhaust the Space	3 (30%)	10 (63%)	8 (35%)
Procedural Tool Use	1 (10%)	0 (0%)	0 (0%)
Quantification	4 (40%)	11 (69%)	6 (26%)

Coding Length Definitions

Of the ten length definitions, three mentioned exhausting the space and four mentioned quantifying the space. Only one mentioned procedural tool use. Three examples are shown in Table 13 to illustrate the concepts. We saw different approaches to quantification. For example, Parker and Baldridge (2008) implied a *partitioning* strategy for quantification, showing a quantity subdivided into unit lengths and describing the process as: "express[ing] other lengths as multiples of that unit [length]" (p. 6). Van de Walle et al. (2013), on the other hand, described quantification as *iterating* (e.g., making identical copies of) the length unit and placing them along the path being measured.

Table 13 shows the different concepts appearing in length definitions. Van de Walle et al. (2013) gave a rationale for exhausting the space, explaining the reasoning behind avoiding gaps because "part of the quantity is not being measured" and overlaps because " part of the quantity is measured more than once" (p. 382). Only Sowder et al. (2010) explicitly mentioned procedural tool use when defining length as "the characteristic of one-dimensional shapes that is measured with a ruler" (p. G-7).

Table 13. Steps 3-4 Length Examples

Definitions (emphasis added)	Length Definition Aspects
Lengths are not numbers because any measurement of length involves a two-step process: Choose a unit length. ***Express other lengths as multiples of that unit.*** *The resulting length is then a quantity; a number times a unit.* (Parker & Baldridge, 2008, p. 6)	Quantification
There are four important principles of iterating units of length… [to] ***accumulate units by counting****. …The units* ***must be without gaps - if not, part of the quantity is not being measured****. The units* ***must not overlap - otherwise, part of the quantity is measured more than once****.* (Van de Walle et al., 2013, p. 382)	Exhaust Space; Quantification
[Length is] the characteristic of one-dimensional shapes that is ***measured with a ruler*** (Sowder et al., 2010, p. G-7)	Procedural Tool Use

Coding Area Definitions

Of the 16 area definitions, most mentioned exhausting the space (10) or quantification (11). The concept of quantification appeared in several forms, ranging from simply counting units to more formal descriptions. Three definitions are shown in Table 14, varying levels of sophistication in quantification.

Table 14. Steps 3-4 Area Examples

Definitions (emphasis added)	Area Definition Aspects
To measure the area of a region informally, we select a convenient two-dimensional shape as our unit and determine ***how many such units are needed to cover the region****.* (Musser et al., 2011, p. 680)	Exhaust Space; Quantification
Area is a way of ***associating to each region R a quantity Area(R) that reflects our intuitive sense of "how big"*** *the region is without reference to the shape of the region. Area is defined [as] ...: 1. Choose a "unit region" and declare its area to be 1 unit of area 2.* ***Express the areas of other regions as multiples of this unit area.*** (Parker & Baldridge, 2008, p. 107-108)	Quantification
We say a ***collection of rectangles {R_j} tile or pave a given rectangle R, if, by combining the R_j's together we get the whole rectangle R, and if the R_j's intersect at most along their boundaries.*** *With all this terminology in place, the area of a general rectangle is by definition the* ***number of unit squares required to pave that rectangle.*** (Wu, 2011, p. 191)	Exhaust Space; Quantification

Musser et al. (2011) described exhausting the space as determining the number of units that cover a region. Parker and Baldridge (2008) refer to area as a function assigning any region to a quantity in a systematic way. Wu (2011) described a more formal way of composing and decomposing the area of a region into the sum of areas of subregions. Wu (2011) defined exhausting the space formally stating that "combining the [smaller rectangles]" results in the whole rectangle being measured (i.e., smaller rectangles cover the rectangle) and " [smaller rectangles] intersect at most along their boundaries" (i.e., leaving no gaps or overlaps) (p. 191).

Coding Volume Definitions

Of the 23 volume definitions, only eight mentioned exhausting the space and only six mentioned quantification. The concept of quantification appeared in several forms, ranging from simply counting units to more formal descriptions. Table 15 shows definitions to illustrate the concepts.

Table 15. Steps 3-4 Volume Examples

Definitions (emphasis added)	Volume Definition Aspects
[The] volume of a solid is the ***number of unit cubes*** *needed to* ***fill*** *the solid.* (Parker & Baldridge, 2008, p. 193)	Exhaust Space; Quantification
[T]he volume of that three-dimensional object is ***how many of a chosen unit of volume*** *(such as cubic inches, cubic centimeters, etc.) it takes to* ***fill the object without gaps or overlaps****, where it is understood that we may use parts of a unit, too.* (Beckmann, 2011, p. 483)	Exhaust Space; Quantification
[V]olume describes how much space a three-dimensional figure contains. The unit of measure for volume must be a shape that tessellates space. Cubes ***tessellates*** *space; that is, they can be* ***stacked so that they leave no gaps and fill space****.* (Billstein et al., 2010, p. 906)	Exhaust Space

The definitions shown in Table 15 illustrate different perspectives on exhausting the space and quantification. Parker and Baldridge (2008) mention only the word "fill," with respect to exhausting the space. Beckmann (2011) begin their definition similarly by mentioning "fill" but then clarify that there can be no "gaps or overlaps" and that fractions of a unit may also be used (p. 483). Billstein et al. (2010) describe more specifically the need for a unit of volume that "tessellates space" and explain that tessellating means "they can be stacked so that they leave no gaps and fill space" (p. 906). In describing quantification, Parker and Baldridge mention the "number of unit cubes" while Beckmann mentions "how many of a chosen unit of volume."

Across Steps and Spatial Measures

In Table 16 below, a summary of coding frequencies across steps and across length, area, and volume/capacity is shown to highlight the wide variety of definitions that were analyzed. Frequencies at or above half are highlighted. Hence, some consistencies are easily seen: almost all definitions mention the concept as a measure or quantification and most mention units. Half of length and area definitions mention dimensionality, and over half of area and volume definitions mention boundaries. Almost half of all definitions mentioned the concept as an attribute in addition to being a measurement. Area definitions seem relatively consistent with each other, with at least half mentioning area as a quantification with dimensionality, boundaries, discrete units including definitions, examples, or rationales, and the need to exhaust the space.

Table 16. Coding Summary across Steps

		Length (10)	Area (16)	Volume/Capacity (23)
Step 1	Concept as attribute, including examples	4 (40%)	7 (44%)	11 (48%)
	Concept as a measure / quantification	9 (90%)	15 (94%)	15 (65%)
	Space is mentioned	–	1 (6%)	11 (48%)
	Dimensionality is stated	5 (50%)	8 (50%)	9 (39%)
	Boundaries, including examples	4 (40%)	8 (50%)	14 (61%)
Step 2	Units mentioned	5 (50%)	13 (81%)	12 (52%)
	Definition, examples, or rationale	2 (20%)	12 (75%)	11 (48%)
	Standard units mentioned	2 (20%)	2 (13%)	11 (48%)
	Nonstandard units mentioned	1 (10%)	2 (13%)	3 (13%)
	Parts or fractions of a unit	1 (10%)	2 (13%)	1 (4%)
	Continuous units	–	2 (13%)	4 (17%)
	Discrete units	4 (40%)	13 (81%)	10 (43%)
Step 3	Exhaust the space	3 (30%)	10 (63%)	8 (35%)
	Procedural tool use	1 (10%)	–	–
Step 4	Quantification	4 (40%)	11 (69%)	6 (26%)

Discussion and Conclusion

Textbooks for future teachers build understanding of concepts such as dimensionality and units through tasks, additional text, and images. The goal in analyzing textbook definitions is to focus attention here on choices that can be made when writing definitions that illustrate particular aspects and alter their level of sophistication, especially in constructing definitions that can support future teachers in making connections and seeing structures. Choices can be made about which structures to focus on; for example, connecting a conceptual and formal set of definitions may help future teachers make connections between spatial reasoning and algebraic reasoning.

That is, to build on Bassarear's (2012) definition of area, one might connect conceptual discussion about length, area, and volume in the following way. *Questions about measuring generally deal with "how much" it takes to connect two endpoints (e.g., length), to cover the surface of an object or the interior of a simple closed curve (e.g., area), to fill (i.e., capacity) or to construct an object or a simple closed surface (i.e., volume).* The more formal use of simple closed curve or figure is included to support future teachers' connections between the real world and geometry, between informal or rough draft language and formal language. Such a description could then be connected to a more formal definition, building on Parker and Baldridge (2008) and Wu (2011). *A measure of an* n-*dimensional space S is a way of associating to each space S a quantity Measure(S) that reflects our intuitive sense of "how big" the space is. To define Measure(S), choose a "unit space" U that is* n-*dimensional and can tessellate the space (i.e., be placed without gaps or overlaps). Declare Measure(U) to be 1. Then the measure of a fraction or other quantity of U is that multiple of 1, i.e., Measure(r·U) =r ·1. Let a collection of copies and fractional copies of U called {U_j}. We say the U_j's exhaust (e.g., connect, cover, or fill) an* n-*dimensional space S, if, by combining the U_j's together we get the whole space S, and if the U_j's intersect at most along their boundaries. Then Measure(S) is the sum of Measure(U_j)'s.*

The concepts of spatial measurement ---length, area, and volume--- seem easy to understand using intuition built on experiences in the real world. The terms seem straightforward, especially to a layperson. The creation of definitions is not a straightforward process, however. In the textbooks examined here, not a single textbook included a complete definition. Beckmann (2011) provided length, area, and volume/capacity definitions that follow a pattern so that their structures can easily be compared. Few textbooks included the meaning of dimensionality or units. Few explained how units are constructed.

Because mathematics textbooks written for future teachers are intended to support them as mathematics learners, most textbooks do not compare or discuss differences in definition types. Sowder et al. (2010) is an example of one textbook that did include more than one definition type that could be compared. They constructed two levels of definitions, one at the level of the future teachers and one at the level of their future students. For example, the future teacher definition of length was "We speak of [length] … in two ways.

The term length might refer to the quality or attribute we are focusing on, or …the measurement of that quality. The context usually makes clear which reference is intended" (p. 528). The student definition of length was "characteristic of one-dimensional shapes that is measured with a ruler; for example, width, height, depth, thickness, perimeter, and circumference all refer to the same characteristic" (p. G-7). Although definitions within few textbooks can be compared, across textbooks differences in approaches and sophistication can easily be seen. For example, most textbook definitions focused on length, area, and volume as measurements without also describing them as attributes. Because English uses the same word for the measurement and the thing being measured, stating this difference in meaning explicitly may be important for future teachers.

One textbook (e.g., Wu, 2011) focused on a formal definition of the measurement of length, area, and volume. The measurement is described as a type of function, assigning only one quantification for a particular space that can be consistently and accurately compared to other spaces as "larger" or "smaller" or "equal." Another textbook (e.g., Parker & Baldridge, 2008) formally defined covering a space and avoid gaps and overlaps. Opportunities to compare formal mathematical language to descriptions using more informal words may help future teachers as they develop their mathematical and professional identities. Giving them opportunities to see the decision-making that goes into constructing definitions could help them think more critically about the language they use and structures they choose to highlight when they translate definitions into explanations for their future students.

Future steps of this research should be to give future K-8 teachers opportunities to engage deliberately with multiple aspects of the definitions. Perhaps future teachers could analyze definitions or tasks, for example, for selected characteristics. Future teachers could also construct their own definitions to explore how different choices may differently support their development of a more robust understanding of measurement, area, and the process of measurement. Researchers can study ways in which future teachers could analyze definitions or tasks, or constructing their own definitions, to gain a deeper understanding on teachers' conception on definitions.

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Appendix

Bassarear, T. (2012). *Mathematics for elementary school teachers (5th ed)*. Belmont, CA: Brooks/Cole-Cengage.

Length: p. 607 - refers to length as a "linear measure"

Area: p. 621 - Questions about area generally deal with "how much" it takes to cover an object - for example, how much fertilizer to cover a lawn, how much material to cover a bed. In order to answer area questions, we have to select an appropriate unit, and thus the answer takes the form of how many of those units.

Surface area: p. 637 - surface area is the area needed to cover all the faces of a three-dimensional object

Volume: p. 608 - Over time, two related systems of measure for what we call volume evolved, depending on whether the thing being measured was dry or wet.

Volume: p. 612 - volume (dry) and capacity (liquid)

Volume: p. 637- the amount of space contained within that object

Volume: p. 641 - volume reflects how much it will take to fill an object

Beckmann, S. (2011). *Mathematics for elementary school teachers (3rd ed).* Boston, MA: Pearson/Addison Wesley.

Length: p. 481 - A length describes the size of something (or a part of something) that is one-dimensional; the length of that one-dimensional object is how many of a chosen unit of length (such as inches, centimeters, etc.) it takes to cover the object without gaps or overlaps, where it is understood that we may use parts of a unit, too. Roughly speaking, an object is one-dimensional if at each location, there is only one independent directions along which to move within the object.

Area: p. 151 - In general, for any unit of length, one square unit is the area of a square that is 1 unit wide and 1 unit long. The area of a region, in square units, is the number of 1-unit-by-1-unit squares it takes to cover the region without gaps or overlaps (where square may be cut apart if necessary)

Surface area: p. 482 - An area or a surface area describes the size of an object (or a part of an object) that is two-dimensional; the area of that two-dimensional object is how many of a chosen unit of area (such as square inches, square centimeters, etc.) it takes to cover the object without gaps or overlaps, where it is understood that we may use parts of a unit, too. The surface area of a solid shape is the total area of its outside surface. Roughly speaking, an object is two-dimensional if, at each location, there are two independent directions along which to move within the object.

Volume: p. 153 - The volume, in cubic units, of a box or box shape is just the number of 1-unit-by-1-unit-by-1-unit cubes that it would take to fill the box or make the box shape (without gaps or overlaps).

Volume: p. 483 - A volume describes the size of an object (or a part of an object) that is three-dimensional; the volume of that three-dimensional object is how many of a chosen unit of volume (such as cubic inches, cubic centimeters, etc.) it takes to fill the object without gaps or overlaps, where it is understood that we may use parts of a unit, too. Roughly speaking, an object is three-dimensional if at each location there are three independent directions along which to move within the object.

Capacity: p. 474 - Capacity is essentially the same as volume, except that the term capacity can be used for the volume of a container, the volume of liquid in a container, or the volume of space taken up by a substance filling a container, such as flour or berries.

Bennett, A.B., Burton, L. J. & Nelson, L. T. (2012). *Math for elementary teachers: A conceptual approach (9th ed).* New York, NY: McGraw-Hill Science/Engineering/Math.
Length: p. 655 - 1 length (or linear) unit
Area: p. 676 - To measure the sizes of plots of land, panes of glass, floors, walls, and other such surfaces, we need a new type of unit, one that can be used to cover a surface. The number of units it takes to cover a surface is called its area.Squares have been found to be the most convenient shape for measuring area.
Surface area: p. 704 - Another important measure associated with objects in space is their amount of surface. Surface area is expressed as the number of unit squares needed to cover the surface of a three-dimensional figure.
Volume: p. 701 - To measure the amount of space in tanks, buildings, refrigerators, cars and other three-dimensional figures, we need units of measure that are also three-dimensional figures. The number of such units needed to fill a figure is its volume. Cubes are convenient because they pack together without gaps or overlapping.

Billstein, R., Libeskind, S., & Lott, J. W. (2010). *A problem solving approach to mathematics for elementary school teachers (10th ed).* Boston, MA: Pearson/Addison-Wesley.
Length: p. 838 - linear measure
Area: p. 854 - quote from "Focal Points": "Students recognize area as an attribute of two-dimensional regions. They learn that they can quantify area by finding the total number of same-sized units of area that cover the shape without gaps or overlaps. They understand that a square unit that is 1 unit on a side is the standard unit for measuring area. "
Surface area: p. 854 - Area is measured using square units and the area of a region is the number of nonoverlapping square units that covers the region.
Volume: p. 906 - volume describes how much space a three-dimensional figure contains. The unit of measure for volume must be a shape that tessellates space. Cubes tessellates space; that is, they can be stacked so that they leave no gaps and fill space. Standard units of volume are based on cubes and are cubic units. A cubic unit is the amount of space enclosed within a cue that measures 1 unit on a side.

Long, C. T., & DeTemple, D. W. (2012). *Mathematical reasoning for elementary teachers (6th ed.).* Boston, MA: Pearson/Addison Wesley.
Area: p. 530 - Area is a measure of the region bounded by a closed plane curve. Any shape could be chosen as a unit, but the square is the most common. The size of the square is arbitrary, but it is natural to choose the length of a side to correspond to a unit measure of length. Areas are therefore usually measured in square inches, square feet, and so on.
Volume: p. 530 - Volume is the measure of space taken up by a solid three-dimensional space. The unit… is the volume of a cube whose side length is one of the standard units of length.
Capacity: p. 531 - Capacity is the volume that can be held in a container such as a bottle, pan, basket, or tank.

Musser, G. L., Burger, W. F., & Peterson, B. E. (2011). *Mathematics for elementary school teachers: A contemporary approach (9th ed.).* New York: John Wiley & Sons, Inc.

Length: p. 680 - Regardless, in every case, we can select some appropriate unit and determine how many units are needed to span the object. This is an informal measurement method of measuring length, since it involves naturally occurring units and is done in a relatively imprecise way.
Area: p. 680 - To measure the area of a region informally, we select a convenient two-dimensional shape as our unit and determine how many such units are needed to cover the region.
Volume: p. 680 - The volume of the vase would be the amount of material comprising the vase itself.
Capacity: p. 680 - To measure the capacity of water that a vase will hold, we can select a convenient container, such as a water glass, to use as our unit and count how many glassfuls are required to fill the vase. This is an informal method of measuring volume. (Strictly speaking, we are measuring the capacity of the vase, namely the amount that it will hold.

Parker, T. H., & Baldridge, S. J. (2008). *Elementary geometry for teachers (Volume 2).* Okemos, MI: Sefton-Ash Publishing.
Length: p. 5 - In geometry one also has a notion of length and distance. The distance between two points A and B is the length of the segment \overline{AB}; in this book we will denote this length by AB (some textbooks use a different notation for length). Note that AB is a number whereas \overline{AB} is a segment.
Length: p. 6 - Lengths are not numbers because any measurement of length involves a two-step process:

1. Choose a unit length. [picture omitted - shows example of 1 unit]
2. Express other lengths as multiples of that unit. [picture omitted - shows multiple of unit "The bar is 4 units long."] The resulting length is then a quantity; a number times a unit. [picture omitted - shows that in 48 inches, 48 is the number and inches is the unit]

Area: p. 107 - 108 - A portion of the plane is called a region. A triangular region is a triangle together with its interior, and a polygonal region is a polygon together with its interior. A circular region is a disk. A region formed from several pieces is a composite region. [image omitted - examples of triangle, triangular region, rectangular region, disk or circular region, composite region] Area is a way of associating to each region R a quantity Area(R) that reflects our intuitive sense of "how big" the region is without reference to the shape of the region. Area is defined by the same two-step scheme used to define length, weight, and capacity:

1. Choose a "unit region" and declare its area to be 1 unit of area
2. Express the areas of other regions as multiples of this unit area.

Area: p. 110 - Definition 1.2 (School Definition). The area of a region tiled by unit squares is the number of squares it contains.
Area: p. 193 - area is the number of unit squares needed to cover a region
Volume: p. 193 - volume of a solid is the number of unit cubes needed to fill the solid
Volume: p. 193 - Volume measurements come in three different forms: liquid, solid, and air/space. Each has unique features that make it easy or hard to measure.

- The volume of a Liquid is easily measured - use a measuring cup.
- The volume of a Solid is easily measured for rectangular solids, but volume is not intuitive or easy to measure for irregular solids.
- The volume of Air/Space is neither intuitive nor easily measured.

Volume: p. 198 - [image provided illustrating the 5 types of measurement that is usually studied in elementary

school and their relationships]
Capacity: p. 15 - The capacity of a container is the amount of liquid it can hold.

Sonnabend, T. (2010). *Mathematics for Elementary Teachers: An Interactive Approach for Grades K-8 (4th ed.).* Belmont, CA: Brooks/Cole-Cengage.
Area: p. 531 - If you want to know the size of the interior of a field, or which package of gift wrap is a better buy, you measure the area. Area is the measure of a closed, two-dimensional region.
Surface area: p. 564: The total surface area of a closed space figure is the sum of the areas of all its surfaces. A surface area in square units indicates how many squares it would take to cover the outside of a space figure.
Volume: p. 571 - Volume is the amount of space occupied by a three-dimensional figure. Volume is usually measured in cubic units. Whereas surface area is the total area of the faces of a solid, volume is the capacity of a solid.
Capacity: p. 391 - The term capacity is generally used to refer to the amount that a container will hold. Standard units of capacity include quarts. gallons, liters, and milliliters. The term volume can be used to refer to the capacity of a container but is also used for the size of solid objects.

Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing Mathematics for Elementary School Teachers: Instructors Edition.* New York: W.H. Freeman and Company.
Length: p. 528 - We speak of the length of a piece of wire or a rectangle in two ways. The term length might refer to the quality or attribute we are focusing on, or it might refer to the measurement of that quality. The context usually makes clear which reference is intended.
Length: p. G-7 - the characteristic of one-dimensional shapes that is measured with a ruler; for example, width, height, depth, thickness, perimeter, and circumference all refer to the same characteristic
Area: p. 547 - We speak of the area of a field, a lake, a country, a geometric shape, a wall, or your body, all of which are examples of surfaces or regions - area is a characteristic of surfaces or regions. As with length, the term area is used to refer both to the attribute ("The wolf wandered over a wide area") and to the measurement ("The area of the field is 15 acres").
Surface area: p. G-2 - the number of square units that would be required to cover the region. the region could be a 2D region, or it could refer to all the surfaces of a 3D figure, in which case it is called the surface area.
Volume: p. 558 - Any real object takes up space. Even a scrap of paper, a speck of dust, or a strand of cobweb takes up some space. The quantity of space occupied is called the volume. As with the terms length and area, the term volume is sometimes used for the attribute as well as the measurement of the attribute.
Volume: p. G-16 - the number of 3D units that could fit inside the region, or that could be used to make an exact model of the region. The units are usually cubic regions (e.g., cm^3).
Capacity: p. 558 - Boxes, cans, and other containers enclose an amount of space; we often refer to their capacities or to the volumes they enclose. The material actually making up a box involves a certain amount of space, so literally, the volume of the box would be the volume that the box would occupy if flattened out. Usually, volume of the container, however, refers to the capacity of the container rather than to the volume of the material that makes up the container.
Capacity: p. G-2 - the volume that a container can hold

Van de Walle, John A., Karp, Karen S. and Bay-Williams, Jennifer M. (2013). *Elementary and Middle School Mathematics: Teaching Developmentally (8th ed.).* New Jersey: Pearson Education, Inc.
Length: p. 382 - There are four important principles of iterating units of length, whether they are nonstandard or standard (Dietiker, Gonulates, Figueras, & Smith, 2010, p. 2): *All units must have equal length - if not, you cannot accumulate units by counting. *All units must be placed on the path being measured - otherwise, a different quantity is being measured. *The units must be without gaps - if not, part of the quantity is not being measured. *The units must not overlap - otherwise, part of the quantity is measured more than once.
Area: p. 384 - Area is the two dimensional space inside a region.
Volume: p. 391 - Volume and capacity are both terms for measures of the "size" of three-dimensional regions...The term volume can be used to refer to the capacity of a container but is also used for the size of solid objects. Standard units of volume are expressed in terms of length units, such as cubic inches or cubic centimeters.

Wu, H. H. (2011). *Understanding Numbers in Elementary School Mathematics.* American Mathematical Society.
Area: p. 45 - We begin with a definition of area that is adequate for the present need. A square with length 1on each side is called a unit square. The area of the unit square is by definition equal to 1. We say a collection or rectangles {R_j} tile or pave a given rectangle R, if, by combining the R_j's together we get the whole rectangle R, and if the R_j's intersect at most along their boundaries. With all this terminology in place, the area of a general rectangle is by definition the number of unit squares required to pave that rectangle. (Remember that we are dealing only with whole numbers at this point and therefore the lengths of all rectangles are whole numbers.)
Area: p. 191 - Because we will have to use the concept of area in a more elaborate fashion, let us first give a more detailed discussion of the basic properties of area. The basic facts about area that we need are rather mundane and are summarized below.

1. The area of a planar region is always a number.
2. The area of the unit square is by definition the number 1.
3. If two regions are congruent, then their areas are equal.
4. If two regions have at most (part of) their boundaries in common, then the area of the region obtained by combining the two is the sum of their individual areas.